Lectures on Electromagnetic Field Theory

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[1] These polarizations are also variously know as the s and p polarizations, a descendent from the notations for acoustic waves where s and p stand for shear and pressure waves respectively.

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Preface

This set of lecture notes is from my teaching of ECE 604, Electromagnetic Field Theory, at ECE, Purdue University, West Lafayette. It is intended for entry level graduate students. Because different universities have different undergraduate requirements in electromagnetic field theory, this is a course intended to "level the playing field". From this point onward, hopefully, all students will have the fundamental background in electromagnetic field theory needed to take advance level courses at Purdue.

In developing this course, I have drawn heavily upon knowledge of our predecessors in this area. Many of the textbooks and papers used, I have listed them in the reference list. Being a practitioner in this field for over 40 years, I have seen electromagnetic theory impacting modern technology development unabated. Despite its age, the set of Maxwell's equations has continued to be important, from statics to optics, from classical to quantum, and from nanometer lengthscales to galactic lengthscales. The applications of electromagnetic technologies have also been tremendous and wide-ranging: from geophysical exploration, remote sensing, bio-sensing, electrical machinery, renewable and clean energy, biomedical engineering, optics and photonics, computer chip and computer system designs and many more. Electromagnetic field theory is not everything, but is remains an important component of modern technology developments.

The challenge in teaching this course is on how to teach over 150 years of knowledge in one semester: Of course this is mission impossible! To do this, we use the traditional wisdom of engineering education: Distill the knowledge, make it as simple as possible, and teach the fundamental big ideas in one short semester. Because of this, you may find the flow of the lectures erratic. Some times, I feel the need to touch on certain big ideas before moving on, resulting in the choppiness of the curriculum.

Also, in this course, I exploit mathematical homomorphism as much as possible to simplify the teaching. After years of practising in this area, I find that some advanced concepts, which may become very complex, if one delves into the details, become simpler if mathematical homomorphism is exploited between the advanced concepts and simpler ones. An example of this is on waves in layered media. The problem is homomorphic to the transmission line problem: Hence, using transmission line theory, one can simplify the derivations of some complicated formulas.

A large part of modern electromagnetic technologies is based on heuristics. This is something difficult to teach, as it relies on physical insight and experience. Modern commercial software has reshaped this landscape, as the field of mathematical modeling through numerical simulations, known as computational electromagnetic (CEM), has made rapid advances in recent years. Many cut-and-try laboratory experiments, based on heuristics, have been replaced by cut-and-try computer experiments, which are a lot cheaper.

An exciting modern development is the role of electromagnetics and Maxwell's equations in quantum technologies. We will connect Maxwell's equations toward the end of this course. This is a challenge, as it has never been done before to my knowledge.

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